Detailed proof of Marsaglia's algorithm

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October 2026

The algorithm

Marsaglia's algorithm generates a point uniformly distributed on the unit sphere as follows:

1. generate V_1 and V_2 uniform on $(-1, 1)$ and compute $S = V_1^2 + V_2^2$;
2. if $S \geq 1$, reject the pair and go back to step 1;
3. return $\big(2V_1(1-S)^{1/2},\ 2V_2(1-S)^{1/2},\ 1-2S\big)$.

The pair (V_1, V_2) is uniform on the square $(-1, 1)^2$, with density $1/4$. The acceptance probability is the ratio between the area of the interior of the unit circle and that of the square, $\pi/4$, hence the accepted pairs are uniform over the interior of the unit circle, with density $(1/4)/(\pi/4) = 1/\pi$. Both proofs below start from this uniform distribution.

Detailed proof

Consider the following three lemmas:

Lemma 1. The area A of a spherical cap of height h on a unit sphere is equal to:

$$A(h) = 2\pi h \tag{1}$$

If (Z_1, Z_2, Z_3) is a uniform vector over the unit sphere, the probability P_A of finding the vector in a region of area A on the unit sphere is equal to:

$$P_A = \frac{A}{4\pi}, \tag{2}$$

hence, the probability $P(Z_i)dZ_i$ of having a value of Z_i in $(Z_i, Z_i + dZ_i)$ can be written as:

$$P(Z_i)dZ_i = \frac{dA}{4\pi} \tag{3}$$

where dA is the area of the zone of the unit sphere between the planes Z_i and $Z_i + dZ_i$ perpendicular to the i-th axis. This zone is the difference between two

caps with axis along the i-th axis, of heights $h + dZ_i$ and $h = 1 + Z_i$, hence from Eq. (1)

$$dA = A(h + dZ_i) - A(h) = 2\pi dZ_i \tag{4}$$

hence by plugging Eq. (4) in Eq. (3) we have

$$P(Z_i)dZ_i = \frac{1}{4\pi} 2\pi dZ_i = \frac{1}{2} dZ_i \tag{5}$$

which means that each Z_i is uniform over $(-1, 1)$.

Moreover, since (Z_1, Z_2, Z_3) is a uniform vector over the unit sphere, given Z_3, the vector (Z_1, Z_2) is uniform over the circumference of the circle of radius $(1 - Z_3^2)^{1/2}$. Indeed, let us use cylindrical coordinates (θ, Z_3) on the unit sphere, where $\theta \in (0, 2\pi)$ is the azimuthal angle of (Z_1, Z_2), i.e.

$$(Z_1, Z_2) = (1 - Z_3^2)^{1/2} (\cos\theta, \sin\theta). \tag{6}$$

The area element of the unit sphere in these coordinates is

$$dA = d\theta \, dZ_3 \tag{7}$$

(the circle at height Z_3 has radius $\rho = (1 - Z_3^2)^{1/2}$, and the strip between Z_3 and $Z_3 + dZ_3$ has width dZ_3/ρ along the sphere, hence $dA = \rho \, d\theta \cdot dZ_3/\rho$; integrating over θ one recovers $dA = 2\pi \, dZ_3$ of Eq. (4)). Hence the probability of having θ in $(\theta, \theta + d\theta)$ and Z_3 in $(Z_3, Z_3 + dZ_3)$ is

$$P(\theta, Z_3) \, d\theta \, dZ_3 = \frac{dA}{4\pi} = \frac{1}{4\pi} \, d\theta \, dZ_3, \tag{8}$$

and, since $P(Z_3) = 1/2$ from the first part of this lemma (Eq. (5) with $i = 3$), the conditional probability density of θ for a given Z_3 is

$$P(\theta | Z_3) = \frac{P(\theta, Z_3)}{P(Z_3)} = \frac{1/(4\pi)}{1/2} = \frac{1}{2\pi}, \tag{9}$$

i.e. θ is uniform over $(0, 2\pi)$ for a given Z_3: the vector (Z_1, Z_2) is uniform over the circumference of radius $(1 - Z_3^2)^{1/2}$.

Lemma 2. Consider a uniform vector (V_1, V_2) over the interior of the unit circle and let $r = (V_1^2 + V_2^2)^{1/2}$, then the point $(V_1/r, V_2/r)$ is a uniform random point on the unit circle. Indeed, if we use polar coordinates the point can be rewritten as $(\cos\theta, \sin\theta)$, where $\theta \in (0, 2\pi)$ is the polar angle of (V_1, V_2), i.e.

$$\theta = \begin{cases} \arccos(V_1/r) & V_2 \geq 0 \\ 2\pi - \arccos(V_1/r) & V_2 < 0 \end{cases} \tag{10}$$

The probability $P(\theta)d\theta$ of having a vector (V_1, V_2) such that one obtains the point $(\cos\theta, \sin\theta)$ on the unit circle in $(\theta, \theta + d\theta)$ is

$$P(\theta)d\theta = \frac{dA}{\pi} \tag{11}$$

where dA is the area of a circular sector corresponding to angle $d\theta$, i.e.:

$$dA = \frac{1}{2} d\theta \tag{12}$$

hence

$$P(\theta) d\theta = \frac{1}{2\pi} d\theta \tag{13}$$

which means that θ is uniform over $(0, 2\pi)$.

Lemma 3. Consider a uniform vector (V_1, V_2) over the interior of the unit circle $\mathcal{C}$, then we have

$$\int_{\mathcal{C}} \frac{1}{\pi} dV_1 dV_2 = 1 \tag{14}$$

if we switch to polar coordinates (r, θ), with $r = (V_1^2 + V_2^2)^{1/2}$, the area element becomes

$$dV_1\, dV_2 = r\, dr\, d\theta = \frac{1}{2}\, d\theta\, d(r^2), \tag{15}$$

since $d(r^2) = 2r\, dr$. Hence the probability of having θ in $(\theta, \theta + d\theta)$ and r^2 in $(r^2, r^2 + d(r^2))$ is

$$P(\theta, r^2)\, d\theta\, d(r^2) = \frac{1}{\pi}\, dV_1\, dV_2 = \frac{1}{2\pi}\, d\theta\, d(r^2), \tag{16}$$

i.e. the probability density is constant on the rectangle $(0, 2\pi) \times (0, 1)$ of the (θ, r^2) plane: $r^2 = V_1^2 + V_2^2$ is uniform on $(0, 1)$. Note also that r^2 is independent of the point $(V_1/r, V_2/r)$, which is a uniform random point over the circumference of the unit circle (see Lemma 2). Indeed, the point $(V_1/r, V_2/r)$ can be written as $(\cos\theta, \sin\theta)$, with θ given by Eq. (10), and Eq. (16) shows that the joint probability density of θ and r^2 factorizes:

$$P(\theta, r^2) = \frac{1}{2\pi} \cdot 1 = P(\theta)\, P(r^2), \tag{17}$$

with $P(\theta) = 1/(2\pi)$ (Lemma 2) and $P(r^2) = 1$, i.e. r^2 and θ are two independent variables.

From Lemma 2, if (V_1, V_2) is a uniform random vector over the interior of the unit circle, $r = (V_1^2 + V_2^2)^{1/2}$ and Z_3 a given number in $(-1, 1)$, then

$$\left(\frac{V_1}{r} (1 - Z_3^2)^{1/2},\ \frac{V_2}{r} (1 - Z_3^2)^{1/2} \right) \tag{18}$$

is a uniform vector over the circumference of radius $(1 - Z_3^2)^{1/2}$. Moreover, the vector

$$\left(\frac{V_1}{r} (1 - Z_3^2)^{1/2},\ \frac{V_2}{r} (1 - Z_3^2)^{1/2},\ Z_3 \right) \tag{19}$$

is a uniform vector on the unit sphere if Z_3 is a uniform number in $(-1, 1)$ independent of the point $(V_1/r, V_2/r)$, i.e. of its polar angle θ, which is also the

azimuthal angle of the vector (19). Lemma 1 proves the opposite implication; this one follows from the area element of Eq. (7). Indeed, in this case the joint probability of θ and Z_3 is

$$P(\theta, Z_3)\, d\theta\, dZ_3 = P(\theta)\, P(Z_3)\, d\theta\, dZ_3 = \frac{1}{2\pi}\,\frac{1}{2}\, d\theta\, dZ_3 = \frac{dA}{4\pi}, \tag{20}$$

with $dA = d\theta\, dZ_3$ given by Eq. (7), i.e. the vector is uniform on the unit sphere.

From Lemma 3, $r^2 = V_1^2 + V_2^2$ is a uniform random number in $(0, 1)$ independent of $(V_1/r, V_2/r)$, hence $1 - 2r^2$ is a uniform random number in $(-1, 1)$ independent of $(V_1/r, V_2/r)$. As a consequence we can replace Z_3 in Eq. (19) with $1 - 2r^2$; since $1 - Z_3^2$ then becomes $4r^2(1 - r^2)$, we obtain

$$\left(\frac{V_1}{r}\, 2r(1-r^2)^{1/2},\ \frac{V_2}{r}\, 2r(1-r^2)^{1/2},\ 1 - 2r^2\right) = \left(2V_1(1-r^2)^{1/2},\ 2V_2(1-r^2)^{1/2},\ 1 - 2r^2\right) \tag{21}$$

which is the Marsaglia formula to generate vectors uniformly distributed on the unit sphere, if we rename r^2 as S.

Compact proof

The proof above can be condensed into two changes of variables and one substitution.

Unit sphere. Let $\theta \in (0, 2\pi)$ be the azimuthal angle of a point (Z_1, Z_2, Z_3) on the unit sphere, i.e. $(Z_1, Z_2) = (1-Z_3^2)^{1/2}(\cos\theta, \sin\theta)$. In cylindrical coordinates the area element is $dA = d\theta\, dZ_3$ (see Eq. (7)), hence the point is uniform on the sphere, i.e. the probability of finding it in a surface element dA is $dA/(4\pi)$, if and only if

$$P(\theta, Z_3)\, d\theta\, dZ_3 = \frac{1}{4\pi}\, d\theta\, dZ_3 = \frac{d\theta}{2\pi}\,\frac{dZ_3}{2}, \tag{22}$$

i.e. if and only if θ and Z_3 are independent and uniform on $(0, 2\pi)$ and $(-1, 1)$ respectively.

Unit disk. Let $(V_1, V_2) = r(\cos\theta, \sin\theta)$ be a uniform point over the interior of the unit circle and $S = r^2 = V_1^2 + V_2^2$. Since $dV_1\, dV_2 = r\, dr\, d\theta = \frac{1}{2}\, d\theta\, dS$, we have

$$P(\theta, S)\, d\theta\, dS = \frac{1}{\pi}\, dV_1\, dV_2 = \frac{d\theta}{2\pi}\, dS, \tag{23}$$

i.e. θ and S are independent and uniform on $(0, 2\pi)$ and $(0, 1)$ respectively.

Substitution. Hence $Z_3 = 1 - 2S$ is uniform on $(-1, 1)$ and independent of θ, and by the first step the point

$$\Big((1 - Z_3^2)^{1/2}\cos\theta,\ (1 - Z_3^2)^{1/2}\sin\theta,\ Z_3\Big), \tag{24}$$

whose azimuthal angle is θ and whose height is Z_3, is uniform on the unit sphere. Since $1 - Z_3^2 = 4S(1 - S)$ and $(\cos\theta, \sin\theta) = (V_1, V_2)/S^{1/2}$, this point is

$$\Big(2V_1(1 - S)^{1/2},\ 2V_2(1 - S)^{1/2},\ 1 - 2S\Big), \tag{25}$$

i.e. Marsaglia's formula.